De-Morgan's and Bertrand's Test.

— let $\sum u_n$ be a series of the terms such that

$$\lim \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = l$$

Then the series is

i) Convergent if $l > 1$

2) divergent if $l < 1$

3) No firm decision is possible if $l = 1$

8) Test for Convergence

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \cdots$$

Solution:

$$u_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots\cdots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots\cdots (2n)^2}$$

$$u_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots\cdots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots\cdots (2n)^2 (2n+2)^2}.$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$= \frac{4n^2 + 8n + 4}{4n^2 + 4n + 1}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$

So, D'Alembert's Ratio Test fails.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+8n+4-4n^2-4n-1}{4n^2+4n+1}$$

$$= \frac{4n+3}{4n^2+4n+1}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{4n+3}{4n^2+4n+1}\right]$$

$$= \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1}$$

$$= 1$$

So, Raabe's Test fails.

Now, $\cancel{\left\{n\left[\frac{u_n}{u_{n+1}} - 1\right] - 1\right\} \log n}$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] - 1 = \frac{4n^2+3n-4n^2-4n-1}{4n^2+4n-1}$$

$$= \frac{-(n+1)}{4n^2+4n+1}$$

Now,

$$\lim_{n\to\infty}\left\{n\left[\frac{u_n}{u_{n+1}}-1\right]-1\right\}\log n$$

$$= \lim_{n\to\infty} \frac{-(n+1)}{4n^2+4n+1} \cdot \log n.$$

$$= \lim_{n\to\infty} \frac{-n\left(1+\frac{1}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)} \cdot \log n$$

$$= \lim_{n\to\infty} \frac{-\left(1+\frac{1}{n}\right)}{4+\frac{4}{n}+\frac{1}{n^2}} \cdot \frac{\log n}{n}$$

$$= 0 < 1 \qquad \left(\because \lim_{n\to\infty} \frac{\log n}{n} = 0\right.$$

So, by De-Morgan's Bertrand's test the given series is divergent.